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ORIGINAL RESEARCH

New existence results on positive solutions of four-point integral type BVPs for coupled multi-term fractional differential equations

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Abstract In this article, we establish some new existence results on positive solutions of a four-point integral boundary value problem for coupled nonlinear multi-term fractional differential equations. Our analysis rely on the well known fixed point theorems. Numerical examples are given to illustrate the main theorems.

Keywords Four-point integral boundary value problem · Multi-term fractional differential system · Non-Carathéodory function · Fixed-point theorem

Mathematics Subject Classification 92D25 · 34A37 · 34K15

Introduction

Fractional differential systems have many applications in modeling of physical and chemical processes and in engineering [3, 14, 19], and have been of great interest recently. In its turn, mathematical aspects of studies on fractional differential systems were discussed by many authors, see the text books [5, 15] and papers [1, 6, 8, 13, 16, 17, 20, 21, 23–26]. A survey concerning the studies on solvability of two-point or four-point boundary value problems for fractional differential systems was given in [11].

In this paper, we discuss the existence of positive solutions of the following four-point integral type boundary value problem for the multi-term fractional differential system

$$\begin{cases} D_{0^+}^{\alpha}u(t)+p(t)f(t,v(t),D_{0^+}^{n}v(t))=0, a.e., t\in(0,1),\\ D_{0^+}^{\beta}v(t)+q(t)g(t,u(t),D_{0^+}^{m}u(t))=0, a.e., t\in(0,1),\\ \lim_{t\to 0}t^{2-\alpha}u(t)-au(\xi)=\int_0^1\phi_1(t,v(t),D_{0^+}^{n}v(t))dt,\\ u(1)-bu(\eta)=\int_0^1\psi_1(t,v(t),D_{0^+}^{n}v(t))dt,\\ \lim_{t\to 0}t^{2-\beta}v(t)-cv(\xi)=\int_0^1\phi_2(t,u(t),D_{0^+}^{m}u(t))dt,\\ v(1)-dv(\eta)=\int_0^1\psi_2(t,u(t),D_{0^+}^{m}u(t))dt, \end{cases} \tag{1}$$

where

(i) $1<\alpha,\beta\le 2$, $\alpha-1<m<\alpha$ and $\beta-1<n<\beta$, $D_{0^+}^{*}$ is the standard Riemann–Liouville differential derivative of order $*>0$ with the starting point 0,
(ii) $0<\xi\le\eta<1$ and $a,b,c,d\ge 0$,
(iii) $p,q:(0,1)\to\mathbb{R}$, p satisfies that there exist numbers k_1,l_1 such that $k_1>-1$, $\alpha-m+l_1>0$, $2+k_1+l_1>0$ and $|p(t)|<t^{k_1}(1-t)^{l_1}$ for $t\in(0,1)$, q satisfies that there exist numbers k_2,l_2 such that $k_2>-1$, $\beta-n+l_2>0$, $2+k_2+l_2>0$ and $|q(t)|<t^{k_2}(1-t)^{l_2}$ for $t\in(0,1)$, with $p(t)\not\equiv 0$ and $q(t)\not\equiv 0$ on (0, 1),
(iv) $f,g,\phi_i,\psi_i:(0,1)\times[0,+\infty)\times\mathbb{R}\to[0,+\infty)$, f is a strong (n,β)-Carathéory function and g is a strong

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(m,α)-Carathéory function with $f(t,0,0)\not\equiv 0$ and $g(t,0,0)\not\equiv 0$ on (0, 1), ϕ_1,ψ_1 are (n,β)-Carathéory functions and ϕ_2,ψ_2 are (m,α)-Carathéory functions.

A pair of functions (x, y) is called a solution of BVP (1) if $x,y\in C^0(0,1]$ and x, y satisfy all equations in (1). We obtain the results on solutions of BVP(1) using Schauder's fixed point theorem in Banach spaces. The salient features of this study are as follows:

(a) the fractional differential equations in (1) are multi-term ones and their nonlinearities depend on the lower order fractional derivatives with order greater than $\alpha-1$ and $\beta-1$;
(b) instead of the condition $u(0)=0, v(0)=0$ we consider integral boundary conditions which are more suitable as $D_{0+}^{\alpha}x(t)=0$ with $\alpha\in(1,2)$ implies $x(t)=ct^{\alpha-1}$ and obviously x is not continuous at $t=0$ while $\lim_{t\to 0}t^{2-\alpha}x(t)$ exists;
(c) BVP(1) is a generalized form of known ones in references [4, 7, 9, 10, 21], the positive solutions of BVP(1) obtained are unbounded (discontinuous at $t=0$) which are different from those ones (continuous on [0,1]) in [1, 8, 23, 24];
(d) this paper is a complement of [11] in which the existence of positive solutions of BVP(1) was studied under the assumptions $m\in(0,\alpha-1]$ and $n\in(0,\beta-1]$ while $m\in(\alpha-1,\alpha), n\in(\beta-1,\beta)$ are supposed in this paper.

The remainder of this paper is arranged as follows: in Sect. 2, we present preliminary results; in Sect. 3, the main results are presented; and two examples are given in Sect. 4 to illustrate the main results.

Preliminary results

For the convenience of readers, we present here the necessary definitions from fixed point theory and fractional calculus theory.

Definition 2.1 [2] Let X be a Banach space. An operator $T:X\to X$ is completely continuous if it is continuous and maps bounded sets into pre-compact sets (or relatively compact sets).

Definition 2.2 [15] The left Riemann–Liouville fractional integral (left forward) of order $\alpha>0$ of a function $f:(0,\infty)\to\mathbb{R}$ is given by

$$I_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)\mathrm{d}s,\quad t>0$$

provided that the right-hand side exists.

Definition 2.3 [15] The left Riemann–Liouville fractional derivative (left farward) of order $\alpha>0$ of a continuous function $f:(0,\infty)\to\mathbb{R}$ is given by

$$D_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t\frac{f(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s,\quad t>0$$

where $n-1<\alpha<n$, provided that the right-hand side exists.

Definition 2.4 $h:(0,1)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is called a $(m,\alpha)-$Carathédory function if it satisfies

(i) $t\to h(t,t^{\alpha-2}x,t^{2+m-\alpha}y)$ is measurable on (0, 1) for all $(x,y)\in\mathbb{R}^2$,
(ii) $(x,y)\to h(t,t^{\alpha-2}x,t^{2+m-\alpha}y)$ is continuous for a.e. $t\in(0,1)$,
(iii) for each $r>0$, there exists nonnegative number M_r such that $|u|,|v|\le r$ imply
$$\left|h\left(t,t^{\alpha-2}x,t^{2+m-\alpha}y\right)\right|\le M_r,\quad a.e.t\quad\in(0,1).$$

Definition 2.5 $h:(0,1)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is called a (m,α)-Carathédory function if it satisfies

(i) $t\to h(t,t^{\alpha-2}x,t^{2+m-\alpha}y)$ is measurable on (0, 1) for all $(x,y)\in\mathbb{R}^2$,
(ii) $(x,y)\to h(t,t^{\alpha-2}x,t^{2+m-\alpha}y)$ is continuous for a.e. $t\in(0,1)$,
(iii) for each $r>0$, there exists nonnegative function $\phi_r\in L^1(0,1)$ such that $|u|,|v|\le r$ imply
$$\left|h\left(t,t^{\alpha-2}x,t^{2+m-\alpha}y\right)\right|\le\phi_r(t),\quad a.e.t\quad\in(0,1).$$

Lemma 2.1 [15] *Let* $n-1\le\alpha<n$, $u\in C^0(0,\infty)\bigcap L^1(0,\infty)$. *Then*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+C_1t^{\alpha-1}+C_2t^{\alpha-2}+\cdots+C_nt^{\alpha-n},$$

where $C_i\in R$, $i=1,2,\ldots n$

Choose

$$X=\left\{x:(0,1]\to\mathbb{R}\begin{array}{l}x,\ D_{0+}^{m}x\in C^0(0,1]\text{ the following limits exist}\\ \lim\limits_{t\to 0}t^{2-\alpha}x(t),\ \lim\limits_{t\to 0}t^{2+m-\alpha}D_{0+}^{m}x(t)\end{array}\right\}$$

with the norm

$$||x||=\|x\|_X=\max\left\{\sup_{t\in(0,1]}t^{2-\alpha}|x(t)|,\ \sup_{t\in(0,1]}t^{2+m-\alpha}|D_{0+}^{m}x(t)|\right\}$$

for $x\in X$. *It is easy to show that X is a real Banach space.*

Choose

$$Y=\left\{y:(0,1]\to\mathbb{R}\begin{array}{l}y,\ D_{0+}^{n}y\in C^0(0,1]\text{ the following limits exist}\\ \lim\limits_{t\to 0}t^{2-\beta}y(t),\ \lim\limits_{t\to 0}t^{2+n-\beta}D_{0+}^{n}y(t)\end{array}\right\}$$

with the norm

$$||y|| = \|y\|_Y = \max\left\{\sup_{t\in(0,1]} t^{2-\beta}|y(t)|,\ \sup_{t\in(0,1]} t^{2+n-\beta}|D_{0^+}^n y(t)|\right\}$$

for $y \in Y$. It is easy to show that Y is a real Banach space.

Thus, $(X \times Y, ||\cdot||)$ is Banach space with the norm defined by

$$||(x,y)|| = \max\{||x|| = ||x||_X,\ ||y|| = ||y||_Y\} \quad \text{for } (x,y) \in X \times Y.$$

For a function $x : (0,1] \to \mathbb{R}$, a number m and a function $F : (0,1) \times \mathbb{R}^2 \to \mathbb{R}$, denote $F_{m,x}(t) = F(t, x(t), D_{0^+}^m x(t))$.

Denote

$$\begin{aligned}
&\mu_1 = a\xi^{\alpha-1}, \quad \upsilon_1 = 1 - a\xi^{\alpha-2}, \quad \omega_1 = 1 - b\eta^{\alpha-1},\\
&\lambda_1 = 1 - b\eta^{\alpha-2}, \quad \Delta = \mu_1\lambda_1 + \upsilon_1\omega_1,\\
&\mu_2 = c\xi^{\beta-1}, \quad \upsilon_2 = 1 - c\xi^{\beta-2}, \quad \omega_2 = 1 - d\eta^{\beta-1},\\
&\lambda_2 = 1 - d\eta^{\beta-2}, \quad \nabla = \mu_2\lambda_2 + \upsilon_2\omega_2.
\end{aligned} \tag{2}$$

Lemma 2.2 (Lemma 2.6 in [11]) *Suppose that $\Delta \neq 0$ and* **(B0)** *$h \in C^0(0,1)$ and there exist $k > -1$ and $l \leq 0$ such that $2 + l + k > 0$ and $|h(t)| \leq t^k(1-t)^l$ for all $t \in (0,1)$.*

Then $x \in X$ is a solution of problem

$$\begin{cases} D^\alpha x(t) + h(t) = 0, 0<t<1,\\ \lim_{t\to 0} t^{2-\alpha}x(t) - ax(\xi) = M,\ x(1) - bx(\eta) = N \end{cases} \tag{3}$$

if and only if $x \in X$ satisfies

$$\begin{aligned}
x(t) = {}&\frac{\upsilon_1 t^{\alpha-1} + \mu_1 t^{\alpha-2}}{\Delta}N + \frac{\omega_1 t^{\alpha-2} - \lambda_1 t^{\alpha-1}}{\Delta}M\\
&- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s + \frac{\upsilon_1 t^{\alpha-1} + \mu_1 t^{\alpha-2}}{\Delta}\\
&\times \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s\\
&- \frac{b\upsilon_1 t^{\alpha-1} + b\mu_1 t^{\alpha-2}}{\Delta}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s\\
&+ \frac{a\lambda_1 t^{\alpha-1} - a\omega_1 t^{\alpha-2}}{\Delta}\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s.
\end{aligned} \tag{4}$$

Lemma 2.3 (Lemma 2.7 in [11]) *Suppose that $\nabla \neq 0$ and* **(B0)** *holds. Then $y \in Y$ is a solution of problem*

$$\begin{cases} D^\beta y(t) + h(t) = 0, 0<t<1,\\ \lim_{t\to 0} t^{2-\beta}y(t) - cy(\xi) = M,\ y(1) - dy(\eta) = N \end{cases} \tag{5}$$

if and only if $y \in Y$ satisfies

$$\begin{aligned}
y(t) = {}&\frac{\upsilon_2 t^{\beta-1} + \mu_2 t^{\beta-2}}{\nabla}N + \frac{\omega_2 t^{\beta-2} - \lambda_2 t^{\beta-1}}{\nabla}M\\
&- \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}h(s)\mathrm{d}s + \frac{\upsilon_2 t^{\beta-1} + \mu_2 t^{\beta-2}}{\nabla}\\
&\times \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}h(s)\mathrm{d}s\\
&- \frac{d\lambda_2 t^{\beta-1} + d\mu_2 t^{\beta-2}}{\nabla}\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}h(s)\mathrm{d}s\\
&+ \frac{c\lambda_2 t^{\beta-1} - c\omega_2 t^{\beta-2}}{\nabla}\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}h(s)\mathrm{d}s.
\end{aligned} \tag{6}$$

Define the operator T on $X \times Y$, for $(x,y) \in X \times Y$, by $T(x,y)(t) = ((T_1 y)(t), (T_2 x)(t))$ with

$$\begin{aligned}
(T_1 y)(t) = {}&\frac{\upsilon_1 t^{\alpha-1} + \mu_1 t^{\alpha-2}}{\Delta}\int_0^1 \psi_{1n,y}(s)\mathrm{d}s + \frac{\omega_1 t^{\alpha-2} - \lambda_1 t^{\alpha-1}}{\Delta}\\
&\times \int_0^1 \phi_{1n,y}(s)\mathrm{d}s\\
&- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,y}(s)\mathrm{d}s + \frac{\upsilon_1 t^{\alpha-1} + \mu_1 t^{\alpha-2}}{\Delta}\\
&\times \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,y}(s)\mathrm{d}s\\
&- \frac{b\upsilon_1 t^{\alpha-1} + b\mu_1 t^{\alpha-2}}{\Delta}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,x}(s)\mathrm{d}s\\
&+ \frac{a\lambda_1 t^{\alpha-1} - a\omega_1 t^{\alpha-2}}{\Delta}\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,y}(s)\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
(T_2 x)(t) = {}&\frac{\upsilon_2 t^{\beta-1} + \mu_2 t^{\beta-2}}{\nabla}\int_0^1 \psi_{2m,x}(s)\mathrm{d}s + \frac{\omega_2 t^{\beta-2} - \lambda_2 t^{\beta-1}}{\nabla}\\
&\times \int_0^1 \phi_{2m,x}(s)\mathrm{d}s\\
&- \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}q(s)g_{m,x}(s)\mathrm{d}s + \frac{\upsilon_2 t^{\beta-1} + \mu_2 t^{\beta-2}}{\nabla}\\
&\times \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}q(s)g_{m,x}(s)\mathrm{d}s\\
&- \frac{d\upsilon_2 t^{\beta-1} + d\mu_2 t^{\beta-2}}{\nabla}\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}q(s)g_{m,x}(s)\mathrm{d}s\\
&+ \frac{c\lambda_2 t^{\beta-1} - c\omega_2 t^{\beta-2}}{\nabla}\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}q(s)g_{m,x}(s)\mathrm{d}s.
\end{aligned}$$

By Lemmas 2.2 and 2.3, we have that $(x,y) \in X \times Y$ is a solution of BVP(8) if and only if $(x,y) \in X \times Y$ is a fixed point of T.

Lemma 2.4 *Suppose that (i)–(iv) defined in Sect. 1 hold, $\Delta \neq 0$ and $\nabla \neq 0$. Then $T : X \times Y \to X \times Y$ is completely continuous.*

Proof We will prove that both T_1 and T_2 are completely continuous. The proof of the completeness of T_1 is divided into four steps and similarly we can prove that T_2 is completely continuous.

Step 1 Suppose that $\alpha-1<m<\alpha$. We prove that both $T_1: Y\to X$ is well defined.

For $y\in Y$, there exits $r>0$ such that

$$\max\left\{\sup_{t\in(0,1]} t^{2-\beta}|y(t)|,\ \sup_{t\in(0,1]} t^{2+n-\beta}|D_{0^+}^n y(t)||\right\}<r.$$

Then (iii) and (iv) imply that there exists a number $M_r>0$ and $\phi_0,\psi_0\in L^1(0,1)$ such that

$$\begin{aligned}|f(t,y(t),D_{0^+}^n y(t)| &= \left|f\left(t,t^{\beta-2}t^{2-\beta}y(t),\right.\right.\\ &\qquad\left.\left. t^{\beta-n-2}t^{2+n-\beta}D_{0^+}^n y(t)\right)\right|\le M_r,\\ |\phi_1(t,y(t),D_{0^+}^n y(t)|\le\phi_0(t),&\ |\psi_1(t,y(t),D_{0^+}^n y(t)|\le\psi_0(t)\end{aligned}\tag{7}$$

for all $t\in(0,1)$. Then

$$\begin{aligned}&\left|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,y}(s)\mathrm{d}s\right|\\ &\quad\le\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{l_1}M_r\mathrm{d}s\\ &\quad\le M_r\int_0^t\frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{k_1}\mathrm{d}s\\ &\quad= M_r t^{\alpha+k_1+l_1}\int_0^1\frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)}w^{k_1}\mathrm{d}w\\ &\quad=\frac{\mathbf{B}(\alpha+l_1,k_1+1)M_r}{\Gamma(\alpha)}t^{\alpha+k_1+l_1},\\ &\left|\int_0^t\frac{(t-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}p(s)f_{n,y}(s)\mathrm{d}s\right|\\ &\quad\le\int_0^t\frac{(t-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}s^{k_1}(1-s)^{l_1}M_r\mathrm{d}s\\ &\quad\le M_r\int_0^t\frac{(t-s)^{\alpha-m+l_1-1}}{\Gamma(\alpha-m)}s^{k_1}\mathrm{d}s\\ &\quad= M_r t^{\alpha-m+k_1+l_1}\int_0^1\frac{(1-w)^{\alpha-m+l_1-1}}{\Gamma(\alpha)}w^{k_1}\mathrm{d}w\\ &\quad=\frac{\mathbf{B}(\alpha-m+l_1,k_1+1)M_r}{\Gamma(\alpha-m)}t^{\alpha-m+k_1+l_1}.\end{aligned}$$

On the other hand, note $D_{0^+}^m t^\mu=\frac{\Gamma(\mu+1)}{\Gamma(\mu+1-m)}t^{\mu-m}$, $\Gamma(0)=\infty$ with $\frac{1}{\Gamma(0)}=0$, we have

$$\begin{aligned}t^{2-\alpha}(T_1y)(t)=&\frac{\upsilon_1t+\mu_1}{\Delta}\int_0^1\psi_{1n,y}(s)\mathrm{d}s+\frac{\omega_1-\lambda_1t}{\Delta}\int_0^1\phi_{1n,y}(s)\mathrm{d}s\\ &-t^{2-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,y}(s)\mathrm{d}s+\frac{\upsilon_1t+\mu_1}{\Delta}\\ &\times\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,y}(s)\mathrm{d}s\\ &-\frac{b\upsilon_1t+b\mu_1}{\Delta}\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,x}(s)\mathrm{d}s\\ &+\frac{a\lambda_1t-a\omega_1}{\Delta}\int_0^\xi\frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,y}(s)\mathrm{d}s,\end{aligned}$$

$$\begin{aligned}&t^{2+m-\alpha}D_{0^+}^m(T_1y)(t)\\ &=\frac{\upsilon_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}t+\mu_1\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-m-1)}}{\Delta}\int_0^1\psi_{1n,y}(s)\mathrm{d}s\\ &+\frac{\omega_1\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-m-1)}-\lambda_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}t}{\Delta}\int_0^1\phi_{1n,y}(s)\mathrm{d}s-t^{2+m-\alpha}\\ &\times\int_0^t\frac{(t-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}p(s)f_{n,y}(s)\mathrm{d}s\\ &+\frac{\upsilon_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}t+\mu_1\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-m-1)}}{\Delta}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,y}(s)\mathrm{d}s\\ &-\frac{b\upsilon_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}t+b\mu_1\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-m-1)}}{\Delta}\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,x}(s)\mathrm{d}s\\ &+\frac{a\lambda_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}t-a\omega\omega_1\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-m-1)}}{\Delta}\int_0^\xi\frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,y}(s)\mathrm{d}s.\end{aligned}$$

It is easy to show that $T_1y\in X$. So $T_1: Y\to X$ is well defined.

Step 2 Suppose that $\alpha-1<m<\alpha$. Prove that T_1 is continuous.

Let $\{y_i\in Y\}$ be a sequence such that $y_i\to y_0$ as $i\to+\infty$ in Y. Then there exists $r>0$ such that

$$\max\left\{\sup_{t\in(0,1]} t^{2-\beta}|y_i(t)|,\ \sup_{t\in(0,1]} t^{2+n-\beta}|D_{0^+}^n y_i(t)|\right\}\le r$$

holds for $i=0,1,2,\ldots$.

Then (iii) and (iv) imply that there exists a number $M_r>0$ and $\phi_0,\psi_0\in L^1(0,1)$ such that

$$\begin{aligned}|f(t,y_i(t),D_{0^+}^n y_i(t)| &= \left|f\left(t,t^{\beta-2}t^{2-\beta}y_i(t),\right.\right.\\ &\qquad\left.\left. t^{\beta-n-2}t^{2+n-\beta}D_{0^+}^n y_i(t)\right)\right|\le M_r,\\ |\phi_1(t,y_i(t),D_{0^+}^n y_i(t)|&\le\phi_0(t),\\ |\psi_1(t,y_i(t),D_{0^+}^n y_i(t)|&\le\psi_0(t)\end{aligned}\tag{8}$$

for all $t\in(0,1)$. By a direct computation, we get $(T_1y_i)(t)$ and $D_{0^+}^m(T_1y_i)(t)$. One sees that

$$t^{2-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,y_i}(s)\mathrm{d}s-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,y_i}(s)\mathrm{d}s\right|$$
$$\le 2t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{l_1}M_r\mathrm{d}s$$
$$=\frac{2M_r\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}t^{2+k_1+l_1}\le\frac{2M_r\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)},$$
$$t^{2+m-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}p(s)f_{n,y_i}(s)\mathrm{d}s\right.$$
$$\left.-\int_0^t \frac{(t-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}p(s)f_{n,y_i}(s)\mathrm{d}s\right|$$
$$\le 2M_rt^{2+m-\alpha}\int_0^t \frac{(t-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}s^{k_1}(1-s)^{l_1}\mathrm{d}s$$
$$=2M_rt^{2+m-\alpha}t^{\alpha-m+k_1+l_1}\int_0^1 \frac{(1-w)^{\alpha-m+l_1-1}}{\Gamma(\alpha-m)}w^{k_1}\mathrm{d}s$$
$$\le\frac{2M_r\mathbf{B}(\alpha-m+l_1,k_1+1)}{\Gamma(\alpha-m)}t^{2+k_1+l_1}$$
$$\le\frac{2M_r\mathbf{B}(\alpha-m+l_1,k_1+1)}{\Gamma(\alpha-m)}.$$

We can show using the dominant convergence theorem that $T_1y_i\to T_1y_0$ as $i\to+\infty$. Then T_1 is continuous.

Now we prove that T_1 maps bounded sets in Y into relatively compact sets in X. Let $\Omega\subset Y$ be a bounded subset. Then there exists $r>0$ such that

$$\max\left\{\sup_{t\in(0,1]}t^{2-\beta}|y(t)|,\ \sup_{t\in(0,1]}t^{2+n-\beta}|D_{0^+}^n y(t)|\right\}\le r$$

holds for all $y\in\Omega$. Then (iii) and (iv) imply that there exists a number $M_r>0$ and $\phi_0,\psi_0\in L^1(0,1)$ such that

$$|f(t,y(t),D_{0^+}^n y(t)|=\left|f\left(t,t^{\beta-2}t^{2-\beta}y(t),\right.\right.$$
$$\left.\left.t^{\beta-n-2}t^{2+n-\beta}D_{0^+}^n y(t)\right)\right|\le M_r,$$
$$|\phi_1(t,y(t),D_{0^+}^n y(t)|\le\phi_0(t),\quad|\psi_1(t,y(t),D_{0^+}^n y(t)|\le\psi_0(t)\tag{9}$$

for all $t\in(0,1)$.

Step 3 Suppose that $\alpha-1<m<\alpha$. Prove that $\{T_1y:y\in\Omega\}$ is a bounded set in X.

Similar to Step 1 and Step 2, we can show that

$$t^{2-\alpha}|(T_1y)(t)|$$
$$\le\frac{|1-a\xi^{\alpha-2}|+|a|\xi^{\alpha-1}}{|\Delta|}\int_0^1\psi_0(s)\mathrm{d}s$$
$$+\frac{|1-b\eta^{\alpha-1}|+|1-b\eta^{\alpha-2}|}{|\Delta|}\int_0^1\phi_0(s)\mathrm{d}s$$
$$+\frac{M_r\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}+\frac{|1-a\xi^{\alpha-2}|+|a|\xi^{\alpha-1}}{|\Delta|}$$
$$\times\frac{2M_r\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}$$
$$+\frac{|b(1-a\xi^{\alpha-2})|+|ab|\xi^{\alpha-1}}{|\Delta|}\eta^{-2-k_1-l_1}\frac{M_r\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}$$
$$+\frac{|a(1-b\eta^{\alpha-2})|+|a(1-b\eta^{\alpha-1})|}{|\Delta|}$$
$$\xi^{-2-k_1-l_1}\frac{M_r\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)},$$
$$t^{2+m-\alpha}D_{0^+}^m(T_1y)(t)$$
$$\le\frac{|1-a\xi^{\alpha-2}|\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}+|a|\xi^{\alpha-1}\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{|\Delta|}$$
$$\times\int_0^1\psi_0(s)\mathrm{d}s$$
$$+\frac{|1-b\eta^{\alpha-1}|\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}+|1-b\eta^{\alpha-2}|\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}}{|\Delta|}\int_0^1\phi_0(s)\mathrm{d}s$$
$$+\frac{M_r\mathbf{B}(\alpha-m+l_1,k_1+1)}{\Gamma(\alpha-m)}$$
$$+\frac{|1-a\xi^{\alpha-2}|\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}+|a|\xi^{\alpha-1}\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{|\Delta|}$$
$$\times\frac{M_r\mathbf{B}(\alpha-m+l_1,k_1+1)}{\Gamma(\alpha-m)}$$
$$+\frac{|b(1-a\xi^{\alpha-2})|\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}+|ab|\xi^{\alpha-1}\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{|\Delta|}\eta^{m-2-k_1-l_1}$$
$$\times\frac{M_r\mathbf{B}(\alpha-m+l_1,k_1+1)}{\Gamma(\alpha-m)}$$
$$+\frac{|a(1-b\eta^{\alpha-2})|\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}+|a(1-b\eta^{\alpha-1})|\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{|\Delta|}$$
$$\times\xi^{m-2-k_1-l_1}\frac{M_r\mathbf{B}(\alpha-m+l_1,k_1+1)}{\Gamma(\alpha-m)}.$$

So T_1 maps bounded sets into bounded sets in X.

Step 4 Suppose that $\alpha-1<m<\alpha$. Prove that $\{T_1y : y \in \Omega\}$ is a relatively compact set in X.

We prove first that both $\{t^{2-\alpha}(T_1y)(t) : y \in \Omega\}$ and $\{t^{2+m-\alpha}D^m_{0^+}(T_1y)(t) : y \in \Omega\}$ are equi-continuous on (0, 1]. By the definition of T_1, it suffices to show that both

$$\left\{t^{2-\alpha}\int_0^t (t-s)^{\alpha-1}p(s)f_{n,y}(s)\mathrm{d}s : y \in \Omega\right\} \text{ and } \left\{t^{2+m-\alpha}\int_0^t (t-s)^{\alpha-m-1}p(s)f_{n,y}(s)\mathrm{d}s : y \in \Omega\right\}$$

are equi-continuous on (0, 1] (we can prove that the other parts of $\{t^{2-\alpha}(T_1y)(t) : y \in \Omega\}$ and $\{t^{2+m-\alpha}D^m_{0^+}(T_1y)(t) : y \in \Omega\}$ are equi-continuous on (0, 1] similar to [1]). Then, we prove that both $\{t^{2-\alpha}(T_1y)(t) : y \in \Omega\}$ and $\{t^{2+m-\alpha}D^n_{0^+}(T_1y)(t) : y \in \Omega\}$ are equi-convergent as $t \to 0$. By the definition of T_1, it suffices to show that both

$$\left\{t^{2-\alpha}\int_0^t (t-s)^{\alpha-1}p(s)f_{n,y}(s)\mathrm{d}s : y \in \Omega\right\} \text{ and } \left\{t^{2+m-\alpha}\int_0^t (t-s)^{\alpha-m-1}p(s)f_{n,y}(s)\mathrm{d}s : y \in \Omega\right\}$$

are equi-convergent as $t \to 0$.

First, let $t_1, t_2 \in [e,f] \subset (0,1]$ with $t_1<t_2$, $0<e<f\le 1$, and $y \in \Omega$. Then we have

$$\begin{aligned}
&\left|t_1^{2-\alpha}\int_0^{t_1}(t_1-s)^{\alpha-1}p(s)f_{n,y}(s)\mathrm{d}s - t_2^{2-\alpha}\int_0^{t_2}(t_2-s)^{\alpha-1}p(s)f_{n,y}(s)\mathrm{d}s\right|\\
&\le |t_1^{2-\alpha}-t_2^{2-\alpha}|\int_0^{t_2}(t_2-s)^{\alpha-1}|p(s)f_{n,y}(s)|\mathrm{d}s\\
&\quad + t_1^{2-\alpha}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|p(s)f_{n,y}(s)|\mathrm{d}s\\
&\quad + t_1^{2-\alpha}\int_0^{t_1}|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}||p(s)f_{n,y}(s)|\mathrm{d}s\\
&\le M_r|t_1^{2-\alpha}-t_2^{2-\alpha}|\int_0^{t_2}(t_2-s)^{\alpha-1}s^{k_1}(1-s)^{l_1}\mathrm{d}s + M_r t_1^{2-\alpha}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}s^{k_1}(1-s)^{l_1}\mathrm{d}s\\
&\quad + M_r t_1^{2-\alpha}\int_0^{t_1}|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|s^{k_1}(1-s)^{l_1}\mathrm{d}s\\
&\le M_r|t_1^{2-\alpha}-t_2^{2-\alpha}|t_2^{\alpha+k_1+l_1}\mathbf{B}(\alpha+l_1,k_1+1) + M_r t_1^{2-\alpha}t_2^{\alpha+l_1+k_1}\int_{\frac{t_1}{t_2}}^1(1-w)^{\alpha+l_1-1}w^{k_1}\mathrm{d}w\\
&\quad + M_r t_2^{2-\alpha}\int_0^1[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]s^{k_1}(1-s)^{l_1}\mathrm{d}s\\
&\to 0 \text{ uniformly in } \Omega \text{ as } t_1 \to t_2 \text{ on } [e,f].
\end{aligned}$$

Second, let $t_1, t_2 \in [e,f] \subset (0,1]$ with $0<e\le t_1<t_2\le f\le 1$ and $y \in \Omega$. Then we have

$$\begin{aligned}
&\left|t_1^{2+m-\alpha}\int_0^{t_1}(t_1-s)^{\alpha-m-1}p(s)f_{n,y}(s)\mathrm{d}s - t_2^{2+m-\alpha}\int_0^{t_2}(t_2-s)^{\alpha-m-1}p(s)f_{n,y}(s)\mathrm{d}s\right|\\
&\le |t_1^{2+m-\alpha}-t_2^{2+m-\alpha}|\int_0^{t_2}(t_2-s)^{\alpha-m-1}|p(s)f_{n,y}(s)|\mathrm{d}s\\
&\quad + t_1^{2+m-\alpha}\int_{t_1}^{t_2}(t_2-s)^{\alpha-m-1}|p(s)f_{n,y}(s)|\mathrm{d}s\\
&\quad + t_1^{2+m-\alpha}\int_0^{t_1}|(t_1-s)^{\alpha-m-1}-(t_2-s)^{\alpha-m-1}||p(s)f_{n,y}(s)|\mathrm{d}s\\
&\le M_r|t_1^{2+m-\alpha}-t_2^{2+m-\alpha}|\int_0^{t_2}(t_2-s)^{\alpha-m-1}s^{k_1}(1-s)^{l_1}\mathrm{d}s\\
&\quad + M_r t_1^{2+m-\alpha}\int_{t_1}^{t_2}(t_2-s)^{\alpha-m-1}s^{k_1}(1-s)^{l_1}\mathrm{d}s\\
&\quad + M_r t_1^{2+m-\alpha}\int_0^{t_1}[(t_1-s)^{\alpha-m-1}-(t_2-s)^{\alpha-m-1}]s^{k_1}(1-s)^{l_1}\mathrm{d}s\\
&\le M_r|t_1^{2+m-\alpha}-t_2^{2+m-\alpha}|t_2^{\alpha-m+k_1+l_1}\mathbf{B}(\alpha-m+l_1,k_1+1)\\
&\quad + M_r t_1^{2+m-\alpha}t_2^{\alpha-m+k_1+l_1}\int_{\frac{t_1}{t_2}}^1(1-w)^{\alpha-m-1+l_1}w^{k_1}\mathrm{d}w\\
&\quad + M_r t_1^{2+m-\alpha}\int_0^{t_1}(t_1-s)^{\alpha-m-1}s^{k_1}(t_1-s)^{l_1}\mathrm{d}s\\
&\quad - M_r t_1^{2+m-\alpha}\int_0^{t_1}(t_2-s)^{\alpha-m-1}s^{k_1}(t_2-s)^{l_1}\mathrm{d}s\\
&\le M_r|t_1^{2+m-\alpha}-t_2^{2+m-\alpha}|t_2^{\alpha-m+k_1+l_1}\mathbf{B}(\alpha-m+l_1,k_1+1)\\
&\quad + M_r\max\{e^{\alpha-m+k_1+l_1},f^{\alpha-m+k_1+l_1}\}\int_{\frac{t_1}{t_2}}^1(1-w)^{\alpha-m-1+l_1}w^{k_1}\mathrm{d}w\\
&\quad + M_r t_1^{2+m-\alpha}t_1^{\alpha-m+k_1+l_1}\int_0^1(1-w)^{\alpha-m+l_1-1}w^{k_1}\mathrm{d}w\\
&\quad - M_r t_1^{2+m-\alpha}t_2^{\alpha-m+k_1+l_1}\int_0^{\frac{t_1}{t_2}}(1-w)^{\alpha-m+l_1-1}w^{k_1}\mathrm{d}w\\
&\le M_r|t_1^{2+m-\alpha}-t_2^{2+m-\alpha}|t_2^{\alpha-m+k_1+l_1}\mathbf{B}(\alpha-m+l_1,k_1+1)\\
&\quad + M_r\max\{e^{\alpha-m+k_1+l_1},f^{\alpha-m+k_1+l_1}\}\int_{\frac{t_1}{t_2}}^1(1-w)^{\alpha-m-1+l_1}w^{k_1}\mathrm{d}w\\
&\quad + M_r t_1^{2+m-\alpha}\left(t_1^{\alpha-m+k_1+l_1}-t_2^{\alpha-m+k_1+l_1}\right)\int_0^1(1-w)^{\alpha-m+l_1-1}w^{k_1}\mathrm{d}w\\
&\quad + M_r t_1^{2+m-\alpha}t_2^{\alpha-m+k_1+l_1}\int_{\frac{t_1}{t_2}}^1(1-w)^{\alpha-m+l_1-1}w^{k_1}\mathrm{d}w\\
&\to 0 \text{ uniformly in } \Omega \text{ as } t_1 \to t_2 \text{ on } [e,f].
\end{aligned}$$

Third, we have

$$\Big|t^{2-\alpha}(T_1y)(t) - \Big(\frac{a\xi^{\alpha-1}}{\Delta}\int_0^1 \psi_{1n,y}(s)\mathrm{d}s + \frac{(1-b\eta^{\alpha-1})}{\Delta}$$

$$\times \int_0^1 \phi_{1n,y}(s)\mathrm{d}s..$$

$$+\frac{a\xi^{\alpha-1}}{\Delta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,y}(s)\mathrm{d}s - \frac{ab\xi^{\alpha-1}}{\Delta}$$

$$\times \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,x}(s)\mathrm{d}s$$

$$-\frac{a(1-b\eta^{\alpha-1})}{\Delta}\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,y}(s)\mathrm{d}s\Big)\Big|$$

$$\leq \frac{|1-a\xi^{\alpha-2}|t}{|\Delta|}\int_0^1 \psi_0\mathrm{d}s + \frac{|1-b\eta^{\alpha-2}|t}{|\Delta|}\int_0^1 \phi_0(s)\mathrm{d}s$$

$$+t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|p(s)f_{n,y}(s)|\mathrm{d}s + \frac{|1-a\xi^{\alpha-2}|t}{|\Delta|}$$

$$\times \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|p(s)f_{n,y}(s)|\mathrm{d}s$$

$$+\frac{|b(1-a\xi^{\alpha-2})|t}{|\Delta|}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}|p(s)f_{n,x}(s)|\mathrm{d}s$$

$$+\frac{|a(1-b\eta^{\alpha-2})|t}{|\Delta|}\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}|p(s)f_{n,y}(s)|\mathrm{d}s$$

$$\leq \frac{|1-a\xi^{\alpha-2}|t}{|\Delta|}\int_0^1 \psi_0\mathrm{d}s + \frac{|1-b\eta^{\alpha-2}|t}{|\Delta|}\int_0^1 \phi_0(s)\mathrm{d}s$$

$$+t^{2+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}M_r + \frac{|1-a\xi^{\alpha-2}|t}{|\Delta|}$$

$$\times\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}M_r$$

$$+\frac{|b(1-a\xi^{\alpha-2})|t}{|\Delta|}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}M_r$$

$$+\frac{|a(1-b\eta^{\alpha-2})|t}{|\Delta|}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}M_r$$

$\to 0$ uniformly on Ω as $t \to 0$.

Fourth, we have

$$\Big|t^{2+m-\alpha}D_{0^+}^m(T_1y)(t) - \Big(\frac{a\xi^{\alpha-1}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-m-1)}}{\Delta}\int_0^1 \psi_{1n,y}(s)\mathrm{d}s$$

$$+\frac{(1-b\eta^{\alpha-1})\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-m-1)}}{\Delta}\int_0^1 \phi_{1n,y}(s)\mathrm{d}s$$

$$+\frac{a\xi^{\alpha-1}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-m-1)}}{\Delta}$$

$$\times\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,y}(s)\mathrm{d}s$$

$$-\frac{ab\xi^{\alpha-1}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-m-1)}}{\Delta}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,x}(s)\mathrm{d}s$$

$$-\frac{a(1-b\eta^{\alpha-1})\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-m-1)}}{\Delta}\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f_{n,y}(s)\mathrm{d}s\Big)\Big|$$

$$\leq \frac{|1-a\xi^{\alpha-2}|\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}t}{|\Delta|}\int_0^1 \psi_0(s)\mathrm{d}s + \frac{|1-b\eta^{\alpha-2}|\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}t}{|\Delta|}$$

$$\times\int_0^1 \phi_0(s)\mathrm{d}s$$

$$+t^{2+m-\alpha}\int_0^t \frac{(t-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}s^{k_1}(1-s)^{l_1}M_r\mathrm{d}s$$

$$+\frac{|1-a\xi^{\alpha-2}|\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}t}{|\Delta|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{l_1}M_r\mathrm{d}s$$

$$+\frac{|b(1-a\xi^{\alpha-2})|\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}t}{|\Delta|}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{l_1}M_r\mathrm{d}s$$

$$+\frac{|a(1-b\eta^{\alpha-2})|\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}t}{|\Delta|}\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{l_1}M_r\mathrm{d}s$$

$$\leq \frac{|1-a\xi^{\alpha-2}|\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}t}{|\Delta|}\int_0^1 \psi_0(s)\mathrm{d}s + \frac{|1-b\eta^{\alpha-2}|\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}t}{|\Delta|}$$

$$\times\int_0^1 \phi_0(s)\mathrm{d}s$$

$$+t^{2+k_1+l_1}\frac{\mathbf{B}(\alpha-m+l_1,k_1+1)}{\Gamma(\alpha-m)}M_r$$

$$+\frac{|1-a\xi^{\alpha-2}|\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}t}{|\Delta|}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}M_r$$

$$+\frac{|b(1-a\xi^{\alpha-2})|\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}t}{|\Delta|}\eta^{\alpha-2}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}M_r$$

$$+\frac{|a(1-b\eta^{\alpha-2})|\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}t}{|\Delta|}\xi^{\alpha-2}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}M_r$$

$\to 0$ uniformly on Ω as $t \to 0$.

Therefore, $T_1\Omega$ is relatively compact.

From above discussion, T_1 is completely continuous. The proof is completed. □

Define

$$G(t,s)=\frac{1}{\Gamma(\alpha)\Delta}\begin{cases}(\upsilon_1 t^{\alpha-1}+\mu_1 t^{\alpha-2})(1-s)^{\alpha-1}\\ +(\lambda_1 a t^{\alpha-1}-\omega_1 a t^{\alpha-2})(\xi-s)^{\alpha-1}\\ -(\upsilon_1 b t^{\alpha-1}+b\mu_1 t^{\alpha-2})(\eta-s)^{\alpha-1} & 0\le s\le \min\{t,\xi\},\\ -(\mu_1\lambda_1+\omega_1\upsilon_1)(t-s)^{\alpha-1}, \\ (\upsilon_1 t^{\alpha-1}+\mu_1 t^{\alpha-2})(1-s)^{\alpha-1}\\ -(\upsilon_1 b t^{\alpha-1}+b\mu_1 t^{\alpha-2})(\eta-s)^{\alpha-1} & \xi<s\le \min\{t,\eta\},\\ -(\mu_1\lambda_1+\omega_1\upsilon_1)(t-s)^{\alpha-1}, \\ (\upsilon_1 t^{\alpha-1}+\mu_1 t^{\alpha-2})(1-s)^{\alpha-1}\\ -(\upsilon_1 b t^{\alpha-1}+b\mu_1 t^{\alpha-2})(\eta-s)^{\alpha-1}, & \max\{t,\xi\}<s\le\eta,\\ (\upsilon_1 t^{\alpha-1}+\mu_1 t^{\alpha-2})(1-s)^{\alpha-1}\\ +(\lambda_1 a t^{\alpha-1}-\omega_1 a t^{\alpha-2})(\xi-s)^{\alpha-1} & t<s\le\xi,\\ -(\upsilon_1 b t^{\alpha-1}+b\mu_1 t^{\alpha-2})(\eta-s)^{\alpha-1}, \\ (\upsilon_1 t^{\alpha-1}+\mu_1 t^{\alpha-2})(1-s)^{\alpha-1}\\ -(\mu_1\lambda_1+\omega_1\upsilon_1)(t-s)^{\alpha-1}, & \eta<s\le t,\\ (\upsilon_1 t^{\alpha-1}+\mu_1 t^{\alpha-2})(1-s)^{\alpha-1}, & \max\{\eta,t\}<s\le 1,\end{cases}$$

and

$$H(t,s)=\frac{1}{\Gamma(\beta)\nabla}\begin{cases}(\upsilon_2 t^{\beta-1}+\mu_2 t^{\beta-2})(1-s)^{\beta-1}\\ +(\lambda_2 c t^{\beta-1}-\omega_2 c t^{\beta-2})(\xi-s)^{\beta-1} & 0\le s\le \min\{t,\xi\},\\ -(\mu_2\lambda_2+\omega_2\upsilon_2)(t-s)^{\beta-1}, \\ (\upsilon_2 t^{\beta-1}+\mu_2 t^{\beta-2})(1-s)^{\beta-1}\\ -(\upsilon_2 \mathrm{d} t^{\beta-1}+d\mu_2 t^{\beta-2})(\eta-s)^{\beta-1} & \xi<s\le \min\{t,\eta\},\\ -(\mu_2\lambda_2+\omega_2\upsilon_2)(t-s)^{\beta-1}, \\ (\upsilon_2 t^{\beta-1}+\mu_2 t^{\beta-2})(1-s)^{\beta-1}\\ -(\upsilon_2 \mathrm{d} t^{\beta-1}+d\mu_2 t^{\beta-2})(\eta-s)^{\beta-1}, & \max\{t,\xi\}<s\le\eta,\\ (\upsilon_2 t^{\beta-1}+\mu_2 t^{\beta-2})(1-s)^{\beta-1}\\ +(\lambda_2 c t^{\beta-1}-\omega_2 c t^{\beta-2})(\xi-s)^{\beta-1} & t<s\le\xi,\\ -(\upsilon_2 \mathrm{d} t^{\beta-1}+d\mu_2 t^{\beta-2})(\eta-s)^{\beta-1}, \\ (\upsilon_2 t^{\beta-1}+\mu_2 t^{\beta-2})(1-s)^{\beta-1}\\ -(\mu_2\lambda_2+\omega_2\upsilon_2)(t-s)^{\beta-1}, & \eta<s\le t,\\ (\upsilon_2 t^{\beta-1}+\mu_2 t^{\beta-2})(1-s)^{\beta-1}, & \max\{\eta,t\}<s\le 1.\end{cases}$$

Now, we rewrite

$$\begin{aligned}(T(x,y))(t)&=((T_1y)(t),(T_2x)(t))\\ &=\left(\frac{\upsilon_1 t^{\alpha-1}+\mu_1 t^{\alpha-2}}{\Delta}\int_0^1\psi_{1n,y}(s)\mathrm{d}s\right.\\ &\quad+\frac{\omega_1 t^{\alpha-2}-\lambda_1 t^{\alpha-1}}{\Delta}\int_0^1\phi_{1n,y}(s)\mathrm{d}s\\ &\quad+\int_0^1 G(t,s)p(s)f_{n,y}(s)\mathrm{d}s,\\ &\quad\frac{\upsilon_2 t^{\beta-1}+\mu_2 t^{\beta-2}}{\nabla}\int_0^1\psi_{2m,x}(s)\mathrm{d}s\\ &\quad+\frac{\omega_2 t^{\beta-2}-\lambda_2 t^{\beta-1}}{\nabla}\int_0^1\phi_{2m,x}(s)\mathrm{d}s\\ &\quad\left.+\int_0^1 H(t,s)g_{m,x}(s)\mathrm{d}s\right).\end{aligned}$$

Lemma 2.5 (Lemma 2.9 in [11]) *Suppose that* $a,b,c,d\ge 0$, *and*

$$\begin{aligned}&\Delta>0,\ \ 0\le a<\frac{1}{\xi^{\alpha-2}(1-\xi)},\ \ 0\le b<\frac{1}{\eta^{\alpha-1}},\\ &\nabla>0,\ \ 0\le c<\frac{1}{\xi^{\beta-2}(1-\xi)},\ \ 0\le d<\frac{1}{\eta^{\beta-1}}.\end{aligned}\tag{10}$$

Then

$$G(t,s)\ge 0 \text{ for all } t,s\in(0,1),\quad H(t,s)\ge\ 0 \text{ for all } t,s\in(0,1).\tag{11}$$

Main results

In this section, we prove existence result on solutions of BVP(1). Let $\mu_i,\upsilon_i,\omega_i,\lambda_i(i=1,2)$ and Δ,∇ be defined by (10). For $\Phi\in L^1(0,1)$, denote $||\Phi||_1=\int_0^1|\Phi(s)|\mathrm{d}s$. The following assumption will be used in the main theorem.

A function $\Phi:[0,\infty)\times[0,\infty)\to[0,\infty)$ is called a bi-increasing function if both $u\to\Phi(u,v)$ and $v\to\Phi(u,v)$ are increasing. We now list the following assumption:

(B1) there exist $\overline{\phi}_i,\overline{\psi}_i\in L^1(0,1)(i=1,2)$ and bi-increasing functions $\Phi,\Psi,\Phi_i,\Psi_i(i=1,2)$ such that

$$\begin{aligned}&\left|f\left(t,\frac{u}{t^{2-\beta}},\frac{v}{t^{2+n-\beta}}\right)\right|\le\Phi(|u|,|v|),t\in(0,1),u,v\in\mathbb{R},\\ &\left|g\left(t,\frac{u}{t^{2-\alpha}},\frac{v}{t^{2+m-\alpha}}\right)\right|\le\Psi(|u|,|v|),t\in(0,1),u,v\in\mathbb{R},\\ &\left|\phi_1\left(t,\frac{u}{t^{2-\beta}},\frac{v}{t^{2+n-\beta}}\right)\right|\le\overline{\phi}_1(t)\Phi_1(|u|,|v|),t\in(0,1),u,v\in\mathbb{R},\\ &\left|\psi_1\left(t,\frac{u}{t^{2-\beta}},\frac{v}{t^{2+n-\beta}}\right)\right|\le\overline{\psi}_1(t)\Psi_1(|u|,|v|),t\in(0,1),u,v\in\mathbb{R},\\ &\left|\phi_2\left(t,\frac{u}{t^{2-\alpha}},\frac{v}{t^{2+m-\alpha}}\right)\right|\le\overline{\phi}_2(t)\Phi_2(|u|,|v|),t\in(0,1),u,v\in\mathbb{R},\\ &\left|\psi_2\left(t,\frac{u}{t^{2-\alpha}},\frac{v}{t^{2+m-\alpha}}\right)\right|\le\overline{\psi}_2(t)\Psi_2(|u|,|v|),t\in(0,1),u,v\in\mathbb{R}.\end{aligned}$$

For ease expression, denote

$$\begin{aligned}M_1&=\left[\frac{\upsilon_1+\mu_1}{\Delta}||+\frac{\upsilon_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}+\mu_1\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-m-1)}}{\Delta}\right]||\overline{\psi}_1||_1,\\ N_1&=\left[\frac{\omega_1+\lambda_1}{\Delta}||+\frac{\omega_1\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-m-1)}+\lambda_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}}{\Delta}\right]||\overline{\phi}_1||_1,\\ Q_1&=\left[1+\frac{\upsilon_1+\mu_1}{\Delta}+\frac{b\upsilon_1+b\mu_1}{\Delta}\eta^{\alpha+k_1+l_1}\right.\\ &\quad\left.+\frac{a\lambda_1+a\omega_1}{\Delta}\xi^{\alpha+k_1+l_1}\right]\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\\ &\quad+\frac{\mathbf{B}(\alpha-m+l_1,k_1+1)}{\Gamma(\alpha-m)}+\frac{\upsilon_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}+\mu_1\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{\Delta}\\ &\quad\times\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\\ &\quad+\frac{b\upsilon_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}+b\mu_1\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{\Delta}\frac{\eta^{\alpha+k_1+l_1}\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\\ &\quad+\frac{a\lambda_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}+a\omega_1\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{\Delta}\frac{\xi^{\alpha+k_1+l_1}\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)},\end{aligned}$$

and

$$M_2 = \left[\frac{\upsilon_2+\mu_2}{\nabla}|| + \frac{\upsilon_2\frac{\Gamma(\beta)}{\Gamma(\beta-n)} + \mu_2\frac{\Gamma(\beta-1)}{\Gamma(\beta-n-1)}}{\nabla}\right]||\overline{\psi}_2||_1,$$

$$N_2 = \left[\frac{\omega_2+\lambda_2}{\nabla}|| + \frac{\omega_2\frac{\Gamma(\beta-1)}{\Gamma(\beta-n-1)} + \lambda_2\frac{\Gamma(\beta)}{\Gamma(\beta-n)}}{\nabla}\right]||\overline{\phi}_2||_1,$$

$$\begin{aligned} Q_2 = &\left[1 + \frac{\upsilon_2+\mu_2}{\nabla} + \frac{c\upsilon_2+d\mu_2}{\nabla}\eta^{\beta+k_2+l_2}\right. \\ &\left.+\frac{c\lambda_2+c\omega_2}{\nabla}\xi^{\beta+k_2+l_2}\right]\frac{\mathbf{B}(\beta+l_2,k_2+1)}{\Gamma(\beta)} \\ &+\frac{\mathbf{B}(\beta-n+l_2,k_2+1)}{\Gamma(\beta-n)} + \frac{\upsilon_2\frac{\Gamma(\beta)}{\Gamma(\beta-n)} + \mu_2\frac{\Gamma(\beta-1)}{|\Gamma(\beta-n-1)|}}{\nabla} \\ &\times\frac{\mathbf{B}(\beta+l_2,k_2+1)}{\Gamma(\beta)} \\ &+\frac{d\upsilon_2\frac{\Gamma(\beta)}{\Gamma(\beta-n)} + d\mu_2\frac{\Gamma(\beta-1)}{|\Gamma(\beta-n-1)|}}{\nabla}\eta^{\beta+k_2+l_2}\frac{\mathbf{B}(\beta+l_2,k_2+1)}{\Gamma(\beta)} \\ &+\frac{c\lambda_2\frac{\Gamma(\beta)}{\Gamma(\beta-n)} + c\omega_2\frac{\Gamma(\beta-1)}{|\Gamma(\beta-n-1)|}}{\nabla}\xi^{\beta+k_2+l_2}\frac{\mathbf{B}(\beta+l_2,k_2+1)}{\Gamma(\beta)}. \end{aligned}$$

Theorem 3.1 *Suppose that (12) holds, (i)–(iv) defined in Sect. 1 and (B1) hold. Then BVP(1) has at least one positive solution if*

$$\begin{aligned} &M_1\Psi_1(r_2,r_2) + N_1\Phi_1(r_2,r_2) + Q_1\Phi(r_2,r_2) \le r_1, \\ &M_2\Psi_2(r_1,r_1) + N_2\Phi_1(r_1,r_1) + Q_2\Psi(r_1,r_1) \le r_2 \end{aligned} \tag{12}$$

has a solution (r_1, r_2) *satisfying* $r_1 > 0, r_2 > 0$.

Proof From Lemmas 2.2 and 2.3, we know that (x, y) is a solution of BVP(1) if and only if (x, y) is a fixed point of T. From Lemma 2.4, $T : X \times Y \to X \times Y$ is completely continuous. By Lemma 2.5 and (i)–(iv), (x, y) is a positive solution if (x, y) is a solution of BVP(1).

To get a fixed point of T, we apply the Schauder's fixed point theorem. We should define a closed convex bounded subset Ω of E such that $T(\Omega) \subseteq \Omega$. It is easy to see that $\Omega = \{(x,y) \in E : ||x|| \le r_1, ||y|| \le r_2\}$ is a closed convex bounded subset Ω of E.

For $(x,y) \in \Omega$, we get $||x|| \le r_1,\ \ ||y|| \le r_2$. Furthermore, we have

$$\begin{aligned} \left|f\left(t,y(t),D_{0^+}^n y(t)\right)\right| &\le \Phi(t^{2-\beta}|y(t)|, t^{2+n-\beta}|D_{0^+}^n y(t)|) \\ &\le \Phi(r_2,r_2), t\in(0,1), \\ \left|g\left(t,x(t),D_{0^+}^m x(t)\right)\right| &\le \Psi(t^{2-\alpha}|x(t)|, t^{2+m-\alpha}|D_{0^+}^m x(t)|) \\ &\le \Psi(r_1,r_1), t\in(0,1), \\ \left|\phi_1\left(t,y(t),D_{0^+}^n y(t)\right)\right| &\le \overline{\phi}_1(t)\Phi_1(r_2,r_2), t\in(0,1), \\ \left|\psi_1\left(t,y(t),D_{0^+}^n y(t)\right)\right| &\le \overline{\psi}_1(t)\Psi_1(r_2,r_2), t\in(0,1), \\ \left|\phi_2\left(t,x(t),D_{0^+}^m x(t)\right)\right| &\le \overline{\phi}_2(t)\Phi_2(r_1,r_1), t\in(0,1), \\ \left|\psi_2\left(t,x(t),D_{0^+}^m x(t)\right)\right| &\le \overline{\psi}_2(t)\Psi_2(r_1,r_1), t\in(0,1). \end{aligned}$$

By the definition of T, we have

$$\begin{aligned} &t^{2-\alpha}|(T_1y)(t)| \\ &\le \frac{\upsilon_1+\mu_1}{\Delta}||\overline{\psi}_1||_1\Psi_1(r_2,r_2) \\ &\quad+\frac{\omega_1+\lambda_1}{\Delta}||\overline{\phi}_1||_1\Phi_1(r_2,r_2) \\ &\quad+t^{2-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{l_1}\mathrm{d}s\Phi(r_2,r_2) \\ &\quad+\frac{\upsilon_1+\mu_1}{\Delta}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{l_1}\mathrm{d}s\Phi(r_2,r_2) \\ &\quad+\frac{b\upsilon_1+b\mu_1}{\Delta}\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{l_1}\mathrm{d}s\Phi(r_2,r_2) \\ &\quad+\frac{a\lambda_1+a\omega_1}{\Delta}\int_0^\xi\frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{l_1}\mathrm{d}s\Phi(r_2,r_2) \\ &\le \frac{\upsilon_1+\mu_1}{\Delta}||\overline{\psi}_1||_1\Psi_1(r_2,r_2) \\ &\quad+\frac{\omega_1+\lambda_1}{\Delta}||\overline{\phi}_1||_1\Phi_1(r_2,r_2) \\ &\quad+\left[1+\frac{\upsilon_1+\mu_1}{\Delta}+\frac{b\upsilon_1+b\mu_1}{\Delta}\eta^{\alpha+k_1+l_1}\right. \\ &\quad\left.+\frac{a\lambda_1+a\omega_1}{\Delta}\xi^{\alpha+k_1+l_1}\right]\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\Phi(r_2,r_2) \end{aligned}$$

and similarly we get

$$\begin{aligned} &t^{2+m-\alpha}|D_{0^+}^m(T_1y)(t)| \\ &\le \frac{\upsilon_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}+\mu_1\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-m-1)}}{\Delta}||\overline{\psi}_1||_1\Psi_1(r_2,r_2) \\ &\quad+\frac{\omega_1\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-m-1)}+\lambda_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}}{\Delta}||\overline{\phi}_1||_1\Phi_1(r_2,r_2) \\ &\quad+t^{2+m-\alpha}\int_0^t\frac{(t-s)^{\alpha-m-1}}{\Gamma(\alpha-m)}s^{k_1}(1-s)^{l_1}\mathrm{d}s\Phi(r_2,r_2) \\ &\quad+\frac{\upsilon_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}+\mu_1\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{\Delta} \\ &\quad\times\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{l_1}\mathrm{d}s\Phi(r_2,r_2) \\ &\quad+\frac{b\upsilon_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}+b\mu_1\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{\Delta} \\ &\quad\times\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{l_1}\mathrm{d}s\Phi(r_2,r_2) \\ &\quad+\frac{a\lambda_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}+a\omega_1\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{\Delta} \end{aligned}$$

$$\times \int_0^{\xi} \frac{(\xi - s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}(1-s)^{l_1} ds \Phi(r_2, r_2)$$
$$\le \frac{\upsilon_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-m)} + \mu_1 \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-m-1)}}{\Delta} ||\overline{\psi}_1||_1 \Psi_1(r_2, r_2)$$
$$+ \frac{\omega_1 \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-m-1)} + \lambda_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}}{\Delta} ||\overline{\phi}_1||_1 \Phi_1(r_2, r_2)$$
$$+ \left[\frac{\mathbf{B}(\alpha - m + l_1, k_1 + 1)}{\Gamma(\alpha - m)} + \frac{\upsilon_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-m)} + \mu_1 \frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{\Delta} \right.$$
$$\times \frac{\mathbf{B}(\alpha + l_1, k_1 + 1)}{\Gamma(\alpha)}$$
$$+ \frac{b\upsilon_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-m)} + b\mu_1 \frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{\Delta} \frac{\eta^{\alpha+k_1+l_1} \mathbf{B}(\alpha + l_1, k_1 + 1)}{\Gamma(\alpha)}$$
$$+ \frac{a\lambda_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-m)} + a\omega_1 \frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{\Delta}$$
$$\left. \frac{\xi^{\alpha+k_1+l_1} \mathbf{B}(\alpha + l_1, k_1 + 1)}{\Gamma(\alpha)} \Gamma(\alpha) \right] \Phi(r_2, r_2).$$

We get

$$||T_1 y|| \le \frac{\upsilon_1 + \mu_1}{\Delta} ||\overline{\psi}_1||_1 \Psi_1(r_2, r_2) + \frac{\omega_1 + \lambda_1}{\Delta} ||\overline{\phi}_1||_1 \Phi_1(r_2, r_2)$$
$$+ \left[1 + \frac{\upsilon_1 + \mu_1}{\Delta} + \frac{b\upsilon_1 + b\mu_1}{\Delta} \eta^{\alpha+k_1+l_1} \right.$$
$$\left. + \frac{a\lambda_1 + a\omega_1}{\Delta} \xi^{\alpha+k_1+l_1} \right] \frac{\mathbf{B}(\alpha + l_1, k_1 + 1)}{\Gamma(\alpha)} \Phi(r_2, r_2)$$
$$+ \frac{\upsilon_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-m)} + \mu_1 \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-m-1)}}{\Delta} ||\overline{\psi}_1||_1 \Psi_1(r_2, r_2)$$
$$+ \frac{\omega_1 \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-m-1)} + \lambda_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}}{\Delta} ||\overline{\phi}_1||_1 \Phi_1(r_2, r_2)$$
$$+ \left[\frac{\mathbf{B}(\alpha - m + l_1, k_1 + 1)}{\Gamma(\alpha - m)} \right.$$
$$+ \frac{\upsilon_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-m)} + \mu_1 \frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{\Delta} \frac{\mathbf{B}(\alpha + l_1, k_1 + 1)}{\Gamma(\alpha)}$$
$$+ \frac{b\upsilon_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-m)} + b\mu_1 \frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{\Delta} \frac{\eta^{\alpha+k_1+l_1} \mathbf{B}(\alpha + l_1, k_1 + 1)}{\Gamma(\alpha)}$$
$$+ \frac{a\lambda_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-m)} + a\omega_1 \frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{\Delta}$$
$$\left. \times \frac{\xi^{\alpha+k_1+l_1} \mathbf{B}(\alpha + l_1, k_1 + 1)}{\Gamma(\alpha)} \Gamma(\alpha) \right] \Phi(r_2, r_2)$$
$$= M_1 \Psi_1(r_2, r_2) + N_1 \Phi_1(r_2, r_2) + Q_1 \Phi(r_2, r_2).$$

Similarly, we get

$$||T_2 x|| \le M_2 \Psi_2(r_1, r_1) + N_2 \Phi_1(r_1, r_1) + Q_2 \Psi(r_1, r_1).$$

Since (13) has positive solution $r_1 > 0, r_2 > 0$, we choose $\Omega = \{(x, y) \in E : ||x|| \le r_1, ||y|| \le r_2\}$. Then we get $T(\Omega) \subset \Omega$. Hence, the Schauder's fixed point theorem implies that T has a fixed point $(x, y) \in \Omega$. So (x, y) is a positive solution of BVP(1).

The proof of Theorem 3.1 is completed. □

Theorem 3.2 *Suppose*

(B2) *there exists* $\overline{\phi}_i, \overline{\psi}_i \in L^1(0,1) (i = 1, 2)$ *and non-negative constants* $M_\Phi, M_\Psi, M_{\Phi_i}, M_{\Psi_i} (i = 1, 2)$ *such that*

$$\left| f\left(t, \frac{u}{t^{2-\beta}}, \frac{v}{t^{2+n-\beta}}\right) \right| \le M_\Phi, t \in (0, 1), u, v \in \mathbb{R},$$
$$\left| g\left(t, \frac{u}{t^{2-\alpha}}, \frac{v}{t^{2+m-\alpha}}\right) \right| \le M_\Psi, t \in (0, 1), u, v \in \mathbb{R},$$
$$\left| \phi_1\left(t, \frac{u}{t^{2-\beta}}, \frac{v}{t^{2+n-\beta}}\right) \right| \le \overline{\phi}_1(t) M_{\Phi_1}, t \in (0, 1), u, v \in \mathbb{R},$$
$$\left| \psi_1\left(t, \frac{u}{t^{2-\beta}}, \frac{v}{t^{2+n-\beta}}\right) \right| \le \overline{\psi}_1(t) M_{\Psi_1}, t \in (0, 1), u, v \in \mathbb{R},$$
$$\left| \phi_2\left(t, \frac{u}{t^{2-\alpha}}, \frac{v}{t^{2+m-\alpha}}\right) \right| \le \overline{\phi}_2(t) M_{\Phi_2}, t \in (0, 1), u, v \in \mathbb{R},$$
$$\left| \psi_2\left(t, \frac{u}{t^{2-\alpha}}, \frac{v}{t^{2+m-\alpha}}\right) \right| \le \overline{\psi}_2(t) M_{\Psi_2}, t \in (0, 1), u, v \in \mathbb{R}.$$

Then BVP(1) has at least one positive solution.

Proof Let $M_i, N_i, Q_i (i = 1, 2)$ be defined in Theorem 3.1. Choose $\Phi(u, v) = M_\Phi$, $\Psi(u, v) = M_\Psi$, $\Phi_i(u, v) = M_{\Phi_i}$ and $\Psi_i(u, v) = M_{\Psi_i} (i = 1, 2)$. We see that (13) has positive solution

$$r_1 = M_1 M_{\Psi_1} + N_1 M_{\Phi_1} + Q_1 M_\Phi,$$
$$r_2 = M_2 M_{\Psi_2} + N_2 M_{\Phi_1} + Q_2 M_\Psi.$$

The results follows from Theorem 3.1 directly. □

Numerical examples

In this section, we present two examples for the illustration of our main result (Theorems 3.1 and 3.2).

Example 4.1 We consider the following boundary value problem

$$\begin{cases} D_{0^+}^{\frac{19}{10}} u(t) + t^{-\frac{1}{10}}(1-t)^{-\frac{17}{20}} f(t, v(t), D_{0^+}^{\frac{39}{40}} v(t)) = 0, & t \in (0, 1), \\ D_{0^+}^{\frac{39}{20}} v(t) + t^{-\frac{1}{10}}(1-t)^{-\frac{13}{20}} g(t, u(t), D_{0^+}^{\frac{19}{20}} u(t)) = 0, & t \in (0, 1), \\ \lim_{t \to 0} t^{\frac{1}{5}} u(t) - \frac{1}{2} u(1/2) = 0, \\ u(1) - \frac{1}{2} u(3/4) = 0, \\ \lim_{t \to 0} t^{\frac{1}{9}} v(t) - \frac{1}{2} v(1/2) = 0. \\ v(1) - \frac{1}{2} v(3/4) = 0, \end{cases} \tag{13}$$

Then

(i) BVP(13) has at least one positive solution if there exists a constant $H > 0$ such that

$$|f(t,t^{-\frac{1}{20}}u,t^{-\frac{41}{40}}v)| \le H,\quad t\in(0,1), u,v\in\mathbb{R},$$

$$|g(t,t^{-\frac{1}{10}}u,t^{-\frac{21}{20}}v)| \le H,\quad t\in(0,1), u,v\in\mathbb{R}.$$

(ii) BVP(13) has at least one positive solution if

$$|f(t,t^{-\frac{1}{20}}u,t^{-\frac{41}{40}}v)| \le H,\ t\in(0,1), u,v\in\mathbb{R},$$

$$|g(t,t^{-\frac{1}{10}}u,t^{-\frac{21}{20}}v)| \le H,\ t\in(0,1), u,v\in\mathbb{R},$$

$$\begin{aligned}\phi_1(t,t^{-\frac{1}{20}}u,t^{-\frac{41}{40}}v) &= \psi_1(t,t^{-\frac{1}{20}}u,t^{-\frac{41}{40}}v)\\ &= \phi_2(t,t^{-\frac{1}{10}}u,t^{-\frac{21}{20}}v)\\ &= \psi_2(t,t^{-\frac{1}{10}}u,t^{-\frac{21}{20}}v)=0,\end{aligned}$$

$$|f(t,t^{-\frac{1}{20}}u,t^{-\frac{41}{40}}v)| \le c_1+b_1|u|^{\epsilon_1}+a_1|v|^{\delta_1},\ c_1,b_1,a_1\ge 0,\ \epsilon_1,\delta_1>0,$$

$$|g(t,t^{-\frac{1}{10}}u,t^{-\frac{21}{20}}v)| \le c_2+b_2|u|^{\sigma_1}+a_2|v|^{\gamma_1},\ c_2,b_2,a_2\ge 0,\ \sigma_1,\gamma_1>0$$

and one of the followings holds:

(a) $\max\{\epsilon_1,\delta_1\}\max\{\sigma_1,\gamma_1\}<1$;
(b) $\max\{\epsilon_1,\delta_1\}\max\{\sigma_1,\gamma_1\}=1$ with $(38.1089b_1)^{1/\sigma_1}$ $34.0678b_2<1$ or $38.1089b_1(34.0678b_2)^{1/\tau_1}<1$
(c) $\max\{\epsilon_1,\delta_1\}\max\{\sigma_1,\gamma_1\}>1$ for sufficiently small b_1,a_1,b_2,a_2.

Proof Corresponding to BVP(1), we have $\alpha=\frac{19}{10},\beta=\frac{39}{20}$, $m=\frac{19}{20}$ and $n=\frac{39}{40}$, $\xi=\frac{1}{2}$, $\eta=\frac{3}{4}$, $a=b=c=d=\frac{1}{2}$ and $\phi_i(t,u,v)=\psi_i(t,u,v)\equiv 0 (i=1,2)$ and $p(t)=t^{-\frac{1}{10}}(1-t)^{-\frac{17}{20}}$, $q(t)=t^{-\frac{1}{10}}(1-t)^{-\frac{13}{20}}$.

It is easy to see that (i)–(iv) hold with $k_1=-\frac{1}{10}=k_2$, and $l_1=-\frac{17}{20}$, $l_2=-\frac{13}{20}$. One sees that $k_1>-1$, $\alpha-m+l_1>0$, $2+k_1+l_1>0$, $k_2>-1$, $\beta-n+l_2>0$, $2+k_2+l_2>0$. Hence, (i)-(iv) defined in Sect. 1 hold.

By direct calculation using Matlab7, we find that

$$\mu_1=\frac{1}{2}\left(\frac{1}{2}\right)^{\frac{9}{10}},\quad \upsilon_1=1-\frac{1}{2}\left(\frac{1}{2}\right)^{-\frac{1}{10}},$$
$$\omega_1=1-\frac{1}{2}\left(\frac{3}{4}\right)^{\frac{9}{10}},\quad \lambda_1=1-\frac{1}{2}\left(\frac{3}{4}\right)^{-\frac{1}{10}},$$
$$\mu_2=\frac{1}{2}\left(\frac{1}{2}\right)^{\frac{19}{20}},\quad \upsilon_2=1-\frac{1}{2}\left(\frac{1}{2}\right)^{-\frac{1}{20}},$$
$$\omega_2=1-\frac{1}{2}\left(\frac{3}{4}\right)^{\frac{19}{20}},\quad \lambda_2=1-\frac{1}{2}\left(\frac{3}{4}\right)^{-\frac{1}{20}}$$

and

$$\Delta=\mu_1\lambda_1+\upsilon_1\omega_1>0,\ \ 0\le a<\frac{1}{\xi^{\alpha-2}(1-\xi)},\ \ 0\le b<\frac{1}{\eta^{\alpha-1}},$$
$$\nabla=\mu_2\lambda_2+\upsilon_2\omega_2>0,\ \ 0\le c<\frac{1}{\xi^{\beta-2}(1-\xi)},\ \ 0\le d<\frac{1}{\eta^{\beta-1}}.$$

(i) By

It follows from Theorem 3.2 that BVP(13) has at least one positive solution.

(ii) One sees that (B1) holds with

$$\Phi(u,v)=c_1+b_1|u|^{\epsilon_1}+a_1v^{\delta_1},$$
$$\Psi(u,v)=c_2+b_2u^{\sigma_1}+a_2v^{\gamma_1},$$
$$\Phi_i(u,v)=\Psi_i(u,v)=0\ (i=1,2)$$

. Furthermore, we have by direct computation (use Mathlab7.0) that

$$\begin{aligned}Q_1 =& \left[1+\frac{\upsilon_1+\mu_1}{\Delta}+\frac{b\upsilon_1+b\mu_1}{\Delta}\eta^{\alpha+k_1+l_1}\right.\\
&\left.+\frac{a\lambda_1+a\omega_1}{\Delta}\xi^{\alpha+k_1+l_1}\right]\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\\
&+\frac{\mathbf{B}(\alpha-m+l_1,k_1+1)}{\Gamma(\alpha-m)}\\
&+\frac{\upsilon_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}+\mu_1\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{\Delta}\\
&\times\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\\
&+\frac{b\upsilon_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}+b\mu_1\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{\Delta}\\
&\times\frac{\eta^{\alpha+k_1+l_1}\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\\
&+\frac{a\lambda_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-m)}+a\omega_1\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-m-1)|}}{\Delta}\\
&\times\frac{\xi^{\alpha+k_1+l_1}\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\\
\simeq& 67.8769\le 68,\end{aligned}$$

and

$$
\begin{aligned}
Q_2 = & \left[1+\frac{\upsilon_2+\mu_2}{\nabla}+\frac{c\upsilon_2+d\mu_2}{\nabla}\eta^{\beta+k_2+l_2}\right.\\
& \left.+\frac{c\lambda_2+c\omega_2}{\nabla}\xi^{\beta+k_2+l_2}\right]\frac{\mathbf{B}(\beta+l_2,k_2+1)}{\Gamma(\beta)}\\
& +\frac{\mathbf{B}(\beta-n+l_2,k_2+1)}{\Gamma(\beta-n)}\\
& +\frac{\upsilon_2\frac{\Gamma(\beta)}{\Gamma(\beta-n)}+\mu_2\frac{\Gamma(\beta-1)}{|\Gamma(\beta-n-1)|}}{\nabla}\\
& \times\frac{\mathbf{B}(\beta+l_2,k_2+1)}{\Gamma(\beta)}\\
& +\frac{d\upsilon_2\frac{\Gamma(\beta)}{\Gamma(\beta-n)}+d\mu_2\frac{\Gamma(\beta-1)}{|\Gamma(\beta-n-1)|}}{\nabla}\\
& \times\frac{\eta^{\beta+k_2+l_2}\mathbf{B}(\beta+l_2,k_2+1)}{\Gamma(\beta)}\\
& +\frac{c\lambda_2\frac{\Gamma(\beta)}{\Gamma(\beta-n)}+c\omega_2\frac{\Gamma(\beta-1)}{|\Gamma(\beta-n-1)|}}{\nabla}\\
& \times\frac{\xi^{\beta+k_2+l_2}\mathbf{B}(\beta+l_2,k_2+1)}{\Gamma(\beta)}\\
& \simeq 56.4653\leq 57.
\end{aligned}
$$

One sees that inequality system (13) has positive solutions if

$$
\begin{aligned}
& 68[c_1+b_1r_2^{\epsilon_1}+a_1r_2^{\delta_1}]\leq r_1,\\
& 57[c_2+b_2r_1^{\sigma_1}+a_2r_1^{\gamma_1}]\leq r_2
\end{aligned}
\tag{14}
$$

has positive solutions. One sees that if

$$
\begin{aligned}
& 68[c_1+(b_1+a_1)r_2^{\max\{\epsilon_1,\delta_1\}}]\leq r_1,\\
& 57[c_2+(b_2+a_2)r_1^{\max\{\sigma_1,\gamma_1\}}]\leq r_2
\end{aligned}
\tag{14$'$}
$$

has positive solution (r_1,r_2) with $r_1>1$, $r_2>1$, then (14) has positive solution $(\max\{1,r_1\},\max\{1,r_2\}$.

(ii)-(a) $\max\{\epsilon_1,\delta_1\}\max\{\sigma_1,\gamma_1\}<1$. It is easy to see that (14) has positive a positive solution (r_1,r_2) with $r_1>0, r_2>0$. It follows from Theorem 3.1 that BVP(13) has at least one solution if one of the followings holds:

(ii)-(b) $\max\{\epsilon_1,\delta_1\}\max\{\sigma_1,\gamma_1\}=1$. One sees that $(14)'$ becomes

$$68[c_1+(b_1+a_1)r_2]\leq r_1,\quad 57[c_2+(b_2+a_2)r_1]\leq r_2.$$

It is easy to see that the latest inequality system holds for sufficiently large $r_1',r_2'>0$ if $68\times 57(a_1+b_1)(a_2+b_2)<1$. Hence (15) has positive solution $(\max\{1,r_1'\},\max\{1,r_2'\}$. Then BVP(1) has positive solution by Theorem 3.1.

(ii)-(c) $\max\{\epsilon_1,\delta_1\}\max\{\sigma_1,\gamma_1\}>1$. By

$$
\begin{aligned}
& \lim_{(a_1,b_1,c_1)\to(0,0,0)}Q_1[c_1+b_1|u|^{\epsilon_1}+a_1v^{\delta_1}]\\
& =\lim_{(a_2,b_2,c_2)\to(0,0,0)}Q_2[c_2+b_2u^{\sigma_1}+a_2v^{\gamma_1}]=0,
\end{aligned}
$$

we know that (15) has positive solution (r_1,r_2) with $r_i>0$. Then Theorem 3.1 implies that BVP(1) has at least one positive solution if a_1,b_1,c_1,a_2,b_2,c_2 are sufficiently small. The proof is completed. $\square$

Example 4.2 We consider the following boundary value problem

$$
\begin{cases}
D_{0^+}^{\frac{19}{10}}u(t)+t^{-\frac{1}{2}}(1-t)^{-\frac{1}{5}}f(t,v(t),D_{0^+}^{\frac{39}{40}}v(t))=0, & t\in(0,1),\\
D_{0^+}^{\frac{39}{20}}v(t)+t^{-\frac{1}{2}}(1-t)^{\frac{1}{10}}g(t,u(t),D_{0^+}^{\frac{19}{20}}u(t))=0, & t\in(0,1),\\
\lim_{t\to 0}t^{\frac{1}{5}}u(t)-\frac{1}{2}u(1/2)=A,\ \ u(1)-\frac{1}{2}u(3/4)=B,\\
\lim_{t\to 0}t^{\frac{1}{9}}v(t)-\frac{1}{2}v(1/2)=C,\ \ v(1)-\frac{1}{2}v(3/4)=D,
\end{cases}
\tag{15}
$$

where

$$
\begin{aligned}
f(t,u,v) &= t^2+\frac{b_1t^{\frac{1}{20}}u^{\epsilon_1}+a_1t^{\frac{41}{40}}v^{\delta_1}}{\sqrt{2}\sqrt{b_1^2t^{\frac{1}{10}}u^{2\epsilon_1}+a_1^2t^{\frac{41}{20}}v^{2\delta_1}+1}},\ 1\ ,\\
& \qquad b_1\geq 0,\ \epsilon_1,\delta_1>0,\\
g(t,u,v) &= 4t^5+\frac{b_2t^{\frac{1}{10}}u^{\sigma_1}+a_2t^{\frac{21}{20}}v^{\gamma_1}}{\sqrt{2}\sqrt{b_2^2t^{\frac{1}{5}}u^{2\sigma_1}+a_2^2t^{\frac{21}{10}}v^{2\gamma_1}+1}},\ a_2,\\
& \qquad b_2\geq 0,\ \sigma_1,\gamma_1>0.
\end{aligned}
$$

Then BVP(15) has at least one positive solution for sufficiently small $a_i,b_i(i=1,2)$.

Proof Corresponding to BVP(1), we have $\alpha=\frac{19}{10},\beta=\frac{39}{20}$, $m=\frac{19}{20}$ and $n=\frac{39}{40}$, $a=b=c=d=\frac{1}{2}$ and $\phi_1(t,u,v)=A,\psi_1(t,u,v)=B,\phi_2(t,u,v)=C,\psi_2(t,u,v)=D$ and $p(t)=t^{-\frac{1}{2}}(1-t)^{-\frac{1}{5}}$, $q(t)=t^{-\frac{1}{2}}(1-t)^{\frac{1}{10}}$.

It is easy to see that (i)–(iv) hold with $k_1=-\frac{1}{10}=k_2$, and $l_1=-\frac{1}{5}$, $l_2=-\frac{1}{10}$. One sees that $k_1>-1$, $\alpha-m+l_1>0$, $2+k_1+l_1>0$, $k_2>-1$, $\beta-n+l_2>0$, $2+k_2+l_2>0$. One sees $m>\alpha-1$, $n>\beta-1$.

Then similar to Example 4.1, we know that BVP(15) has at least one positive solution by Theorem 3.2. $\square$

Conclusions

In this paper, we establish sufficient conditions for the existence of positive solutions of four-point integral type boundary value problems for singular fractional differential systems. We allow the nonlinearities $p(t)f(t, x, y)$ and $q(t)g(t, x, y)$ in fractional differential equations to be singular at $t = 0$. Both f and g may be super-linear and sub-linear. The analysis relies on some well known fixed point theorems. This paper contributes within the domain of fractional differential equations. The methods can be applied to solve other kinds of four-point integral type boundary value problems for singular fractional differential systems.

In [12, 22], authors studied the existence of positive solutions of two-point boundary value problems for fractional order elastic beam equations. One can discuss the following boundary value problem for nonlinear singular coupled fractional order elastic beam equations of the form

$$\begin{cases} D_{0^+}^{\alpha}u(t) = f(t, v(t), v'(t), v''(t)),\ t \in (0,1), \\ D_{0^+}^{\beta}v(t) = g(t.u(t), u'(t), u''(t)),\ t \in (0,1), \\ \lim_{t\to 0} t^{4-\alpha}u(t) = \lim_{t\to 0} t^{4-\alpha}u'(t) = 0, \\ u(1) = u'(1) = 0, \\ \lim_{t\to 0} t^{4-\beta}v(t) = \lim_{t\to 0} t^{4-\beta}v'(t) = 0, \\ v(1) = v'(1) = 0, \end{cases} \tag{16}$$

where $3 < \alpha, \beta \leq 4$, $D_{0^+}^{*}$ (D^* for short) is the Riemann–Liouville fractional derivative of order $*$, and $f, g : (0,1) \times [0,\infty) \times \mathbb{R}^2 \to [0,\infty)$ is continuous. f, g depend on the lower order fractional derivatives u', v' and u'', v'' and may be singular at $t = 0$ and $t = 1$, f, g are non-Carathéodory functions.

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